NASH EQUILIBRIUM WITHOUT COMMONLY KNOWN CONJECTURES AND MUTUAL KNOWLEDGE OF RATIONALITY[1]

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In this paper we provide epistemic conditions for Nash equilibrium, which are strictly weaker than the standard conditions by Aumann and Brandenburger (1995). We simultaneously substitute (i) common knowledge of conjectures with — the strictly weaker — circular pairwise common knowledge of the conjectures, and (ii) mutual knowledge of rationality with — the strictly weaker — connected pairwise mutual knowledge of rationality. We also show that, unlike the Aumann-Brandenburger conditions, ours do not imply common knowledge of rationality. In fact, we show that they do not even imply mutual knowledge of rationality. We find the latter quite surprising, as this is the first paper in the literature providing sufficient conditions for Nash equilibrium without mutual knowledge of rationality.

Keywords: Nash equilibrium, pairwise common knowledge, pairwise mutual knowledge, rationality, conjectures, epistemic game theory.

1. INTRODUCTION

In their seminal paper, Aumann and Brandenburger (1995) provide epistemic conditions for Nash equilibrium. They showed that in the existence of a common prior, mutual knowledge of rationality and common knowledge of each player's conjectures (about the strategies of the opponents) lead to Nash equilibrium in normal form games with more than two players. As Adam Brandenburger eloquently points out in another paper paper that appeared more or less at the same time (Brandenburger, 1992), the epistemic conditions for Nash equilibrium *contain a surprise: common knowledge (of rationality) is not needed*. And he goes on to state that *common knowledge finally enters the picture, but the connection that emerges is unexpected: it is common knowledge of the conjectures, not the players' rationality, that matters for Nash equilibrium*. This result has attracted a lot of attention, as for the first time the widespread view — that common knowledge of rationality is essential for Nash equilibrium — was challenged.

In a subsequent paper, Polak (1999) showed that Aumann-Brandenburger's conditions in fact imply common knowledge of rationality, thus restoring some of the initial confidence in the importance of common knowledge of rationality for Nash equilibrium. Note that his result heavily relies on the conjectures being commonly known, and rationality being mutual knowledge.

[1]We are indebted to...

In this paper, we provide weaker (than Aumann and Brandenburger (1995)) epistemic conditions for Nash equilibrium, in that we simultaneously relax their two main assumption. On the one hand, we substitute common knowledge of the conjectures with circular pairwise common knowledge of the conjectures, which states that there is a (Hamiltonian) cycle of all players such that each player's conjectures are pairwise commonly known only between himself and his neighbors. At the same time we substitute mutual knowledge of rationality with connected pairwise mutual knowledge of rationality, which states that there is a connected graph such that each player's rationality is mutual knowledge only between himself and his neighbors. Then, we show that these weaker conditions still suffice for Nash equilibrium.

The importance of our main result is twofold:

- First, we provide local conditions for Nash equilibrium. We find these conditions much more tractable than the global standard ones, as they only restrict interactive knowledge among small groups of players and still suffice for a Nash equilibrium. This is particularly important for large games, such as economies with many agents.
- Second, we illustrate by means of an example that our conditions do not imply common knowledge of rationality. In fact, they do not even imply mutual knowledge of rationality, meaning that it is possible under our conditions that a player attaches positive probability to a (non-neighboring) opponent being irrational. We find this result very surprising, as this is the first result to our knowledge that provides epistemic conditions for Nash equilibrium without (even) mutual knowledge of rationality. Moreover, we verify Aumann-Brandenburger's idea according to which common knowledge of rationality is not that crucial for Nash equilibrium.

2. PRELIMINARIES

2.1. *Games*

Let $\Gamma = (I, (S_i)_{i\in I}, (U_i)_{i\in I})$ be game in normal form, where $I = \{1, \ldots, N\}$ denotes the finite set of players with typical element i, and S_i denotes the finite set of pure strategies, also called choices, with typical element s_i for every player $i \in I$. Moreover, define $S := \times_{i\in I} S_i$ with typical element $s = (s_1, \ldots, s_N)$ and $S_{-i} := \times_{j\in I\setminus\{i\}} S_j$ with typical element $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$. Then, the function $U_i : S_i \times S_{-i} \to \mathbb{R}$ denotes player i's utility function.

A probability measure $\mu_i \in \Delta(S_{-i})$ on the set of opponents' choice combinations is called a conjecture of i, where $\mu_i(s_{-i})$ signifies the probability that i attributes to the opponents playing s_{-i}. Slightly abusing notation, let $\mu_i(s_j) := \operatorname{marg}_{S_j} \mu_i(s_j)$ denote the probability that i assigns to j playing s_j. Note that it is standard to admit correlated beliefs, i.e. μ_i is not necessarily a

product measure, hence the probability $\mu_i(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ can differ from the product $\mu_i(s_1) \cdots \mu_i(s_{i-1})\mu_i(s_{i+1}) \cdots \mu_i(s_N)$ of the marginal probabilities.[1] Given conjecture $\mu_i \in \Delta(S_{-i})$, player i's expected utility from playing some strategy $s_i \in S_i$ is given by

$$u_i(s_i, \mu_i) := \sum_{s_{-i} \in S_{-i}} \mu_i(s_{-i}) U_i(s_i, s_{-i}).$$

We say that a strategy s_i is a best response to μ_i, and write $s_i \in BR_i(\mu_i)$, whenever $u_i(s_i, \mu_i) \geq u_i(s_i', \mu_i)$ for all $s_i' \in S_i$.

A randomization over a player's pure strategies is called mixed strategy, and is typically denoted by $\sigma_i \in \Delta(S_i)$ for all $i \in I$. Let $\Delta(S_1) \times \cdots \times \Delta(S_N)$ denote the space of mixed strategy profiles, with typical element $(\sigma_1, \ldots, \sigma_N)$. Slightly abusing terminology, we say that a pure strategy $s_i \in S_i$ is a best response to σ, and write $s_i \in BR_i(\sigma)$, whenever s_i is a best response to the product measure $\sigma_{-i} := \text{marg}_{S_{-i}} \sigma$, which is an element of $\Delta(S_1) \times \cdots \times \Delta(S_{i-1}) \times \Delta(S_{i+1}) \times \cdots \times \Delta(S_N) \subseteq \Delta(S_{-i})$. Nash's (1950) notion of equilibrium can then be defined as follows: a mixed strategy profile $(\sigma_1, \ldots, \sigma_N)$ is a Nash equilibrium of the game Γ, whenever $s_i \in BR_i(\sigma)$ for all $s_i \in supp(\sigma_i)$ and for all $i \in I$.

2.2. *Epistemic Models*

The epistemic approach to game theory analyzes the relation between knowledge, belief, and choice of rational players. While classical game theory is based on the two basic primitives – game form and choice – epistemic game theory adds an epistemic framework as a third elementary component such that knowledge and beliefs can be explicitly modelled in games.

Here, we follow Aumann's approach to epistemic game theory and use the partitional model introduced by Aumann (1976). Formally, an epistemic model of some game Γ is a tuple $\mathcal{A}^\Gamma = \big(\Omega, \pi, (\mathcal{P}_i)_{i \in I}, (\hat{s}_i)_{i \in I}\big)$, also called Aumann model of Γ, consisting of a finite set Ω of possible worlds, also called states, with typical element ω, together with a full support common prior $\pi \in \Delta(\Omega)$. Furthermore, every player $i \in I$ is endowed with an information partition $\mathcal{P}_i$ of Ω, as well as a choice function $\hat{s}_i : \Omega \to S_i$. The cell of $\mathcal{P}_i$ containing the world ω is denoted by $P_i(\omega)$ and contains all worlds considered possible by i at ω. Besides, a set $E \subseteq \Omega$ of possible worlds is called event. Knowledge is formalized in terms of events: the set of states at which agent i knows E is defined as

$$K_i(E) := \{\omega \in \Omega : P_i(\omega) \subseteq E\}.$$

[1] Intuitively, a player's belief on his opponents can be correlated, even though players choose independently from each other.

Then, it is said that i knows E at ω, whenever $\omega \in K_i(E)$. For every player $i \in I$ his choice function $\hat{s}_i$ specifies his pure strategy at each world and is assumed to be $\mathcal{P}_i$-measurable, i.e. $\hat{s}_i(\omega') = \hat{s}_i(\omega)$ for all $\omega' \in P_i(\omega)$, which implies that i knows his own strategy. Note that $\hat{s}_i$ induces a coarsening of $\mathcal{P}_i$ consisting of events of the form $[s_i] := \{\omega \in \Omega : \hat{s}_i(\omega) = s_i\}$.

An event is mutually known if everyone knows it. Formally, $E \subseteq \Omega$ is called mutual knowledge at ω, whenever $\omega \in K(E)$, where

$$K(E) := \bigcap_{i \in I} K_i(E).$$

Iterating the mutual knowledge operator then yields higher-order mutual knowledge. Formally, m-order mutual knowledge of E is inductively defined by $K^m(E) := K(K^{m-1}(E))$ for all $m > 0$ with $K^1(E) := K(E)$. Then, an event E is commonly known whenever everyone knows E, everyone knows that everyone knows E, etc. Formally, common knowledge of E can then be stated as

$$CK(E) := \bigcap_{m > 0} K^m(E).$$

Aumann (1976) introduced an alternative yet formally equivalent definition of common knowledge in terms of the finest common coarsening of the players' information partitions, also called the meet. Formally, let $\mathcal{M} := \mathcal{P}_1 \wedge \cdots \wedge \mathcal{P}_N$ denote the meet, with $M(\omega)$ being the element of $\mathcal{M}$ that contains ω. It can be shown that

$$CK(E) = \{\omega \in \Omega : M(\omega) \subseteq E\}.$$

Given an epistemic model $\mathcal{A}^\Gamma$, for every player $i \in I$ conjectures about the opponents' choices can be derived at every world $\omega \in \Omega$ from the common prior. Formally, for every world $\omega \in \Omega$ and for every $s_{-i} \in S_{-i}$, the probability i's conjecture assigns to s_{-i} is defined as

$$\hat{\mu}_i(\omega)(s_{-i}) := \pi\big([s_{-i}] \mid P_i(\omega)\big),$$

where $[s_{-i}] := [s_1] \cap \cdots \cap [s_{i-1}] \cap [s_{i+1}] \cap \cdots \cap [s_n]$. Note that $\hat{\mu}_i$ is $\mathcal{P}_i$-measurable, i.e. $\hat{\mu}_i(\omega') = \hat{\mu}_i(\omega)$ for all $\omega' \in P_i(\omega)$. Henceforth, let

$$[\mu_i] := \{\omega \in \Omega : \hat{\mu}_i(\omega) = \mu_i\}$$

denote the event that player i entertains conjecture μ_i, and

$$[(\mu_1, \ldots, \mu_N)] := [\mu_1] \cap \cdots \cap [\mu_N].$$

Furthermore, player i is rational at world ω, whenever $\hat{s}_i(\omega) \in BR_i\big(\hat{\mu}_i(\omega)\big)$. Let

$$R_i := \{\omega \in \Omega : \hat{s}_i(\omega) \in BR_i\big(\hat{\mu}_i(\omega)\big)\}$$

denote the event that i is rational. Rationality of all players is then given by the event

$$R := \bigcap_{i \in I} R_i.$$

2.3. *Aumann and Brandenburger's Epistemic Conditions for Nash equilibrium*

In their seminal paper, Aumann and Brandenburger (1995) provide epistemic conditions for Nash equilibrium.[2] Accordingly, if conjectures derived from a common prior are commonly known, and rationality as well as the utility functions are mutual knowledge, then all players different from i entertain the same marginal conjecture about i's choice, and the marginal conjectures constitute a Nash equilibrium of the game. Fixing common knowledge of the game as an implicit background assumption, Aumann and Brandenburger's epistemic foundation for Nash equilibrium can formally be stated as follows.

THEOREM 1 (Aumann and Brandenburger (1995)) *Let Γ be a game and $\mathcal{A}^\Gamma$ be an epistemic model of it. Suppose that at some world $\omega \in \Omega$ there exists a tuple $(\mu_1, \ldots, \mu_N)$ of conjectures such that*

$$\omega \in K(R) \cap CK\big([(\mu_1, \ldots, \mu_N)]\big).$$

Then, there exists a mixed strategy profile $(\sigma_1, \ldots, \sigma_N)$ such that

(i) $\operatorname{marg}_{S_i} \mu_j = \sigma_i$ *for all* $j \in I \setminus \{i\}$,

(ii) $(\sigma_1, \ldots, \sigma_N)$ *is a Nash equilibrium of* Γ.

Subsequently, Polak (1999) shows that common knowledge of conjectures and mutual knowledge of rationality imply common knowledge of rationality. In the context of Theorem 1, Polak's result implies that sufficient conditions for Nash equilibrium without common knowledge of rationality need to relax common knowledge of conjectures or mutual knowledge of rationality. In fact, we will weaken both assumptions in Section 4 below and yet obtain Nash equilibrium.

2.4. *Pairwise interactive knowledge*

The standard intuitive explanation for the emergence of common knowledge is based on public announcement. Accordingly, once an event is publicly announced it becomes common knowledge in the sense that not only everyone knows it, but also everyone knows that everyone knows it, etc. Note

[2]Note that Aumann and Brandenburger (1995) employ the formalism of type structures, which is notationally distinct yet formally equivalent to Aumann models.

that for mutual knowledge to obtain, the agents are only required to each know the event, and hence mere private announcements suffice.

Yet, an event may be publicly (privately) announced to some but *not all* players. For instance, an event could be publicly (privately) announced to *Alice* and *Bob*, but not to *Claire*. Common knowledge (mutual knowledge) of the event between *Alice* and *Bob* would then emerge, but not necessarily common knowledge (mutual knowledge). Due to such epistemic possibilities we now introduce pairwise interactive knowledge operators.

Let $E \subseteq \Omega$ be some event and $i, j \in I$ be two players. We say that E is *pairwise mutual knowledge* between i and j whenever they both know E. Formally, pairwise mutual knowledge of E between i and j is denoted by the event

$$K_{i,j}(E) := K_i(E) \cap K_j(E).$$

Note that mutual knowledge implies pairwise mutual knowledge, but not conversely. We say that E is *pairwise common knowledge* between i and j whenever E is commonly known between them. Formally, let $\mathcal{M}_{i,j} := \mathcal{P}_i \wedge \mathcal{P}_j$ with $M_{i,j}(\omega)$ denoting the element of $\mathcal{M}_{i,j}$ that contains ω. Pairwise common knowledge of E between i and j is then defined as the event

$$CK_{i,j}(E) := \{\omega \in \Omega : M_{i,j}(\omega) \subseteq E\}.$$

Observe that, as $\mathcal{M}_{i,j}$ is a refinement of $\mathcal{M}$, common knowledge implies pairwise common knowledge, but not conversely.

We next define the notions of *connected pairwise common knowledge of rationality* and *circular pairwise common knowledge of conjectures*. In order to do so, let us first provide some necessary definitions.

DEFINITION 1 A finite sequence of players $\{i_k\}_{k=1}^m$ is *connected*, if for every $i \in I$ there exists some $k \in \{1, \ldots, m\}$ such that $i_k = i$.

DEFINITION 2 A finite sequence of players $\{i_k\}_{k=1}^{m+1}$ is *strongly connected*, if it is connected and also $i_{m+1} = i_1$.

DEFINITION 3 We say that *rationality is connected pairwise mutual knowledge* at ω whenever there is a connected sequence sequence of players $\{i_k\}_{k=1}^m$, such that

$$\omega \in \bigcap_{k=1}^{m-1} K_{i_k,i_{k+1}}\big(R_{i_k} \cap R_{i_{k+1}}\big).$$

DEFINITION 4 The *conjectures are circular pairwise common knowledge* at ω whenever there is a strongly connected sequence of players $\{i_k\}_{k=1}^{m+1}$, such that

$$\omega \in \bigcap_{k=1}^{m} CK_{i_k,i_{k+1}}\big([\mu_{i_k}] \cap [\mu_{i_{k+1}}]\big).$$

In fact, the preceding two conditions simply place epistemic conditions on the agents' interactive knowledge, and in principle they do not represent physical connectedness of the agents, although they can be interpreted as a network structure. For instance, connected pairwise knowledge between two players might be interpreted as following from the fact that connected players entertain a closer relationship than non-connected ones, and hence share more knowledge. Thus, our restrictions can be illustrated by means of a graph, in which the vertices depict the players and the edges signify the symmetric pairwise common (mutual) knowledge relations.

3. MOTIVATING EXAMPLE

In this section, we provide an example of a case where circular pairwise common knowledge of conjectures and connected pairwise mutual knowledge of rationality lead to Nash equilibrium. The interest in this particular example stems from the fact that the conjectures are not even mutual known, not to mention commonly known. Moreover, rationality is not even mutual knowledge, again not to mention common knowledge. The last two points imply that the standard conditions of Aumann and Brandenburger (1995) are not satisfied, and still we obtain Nash equilibrium. Of course, in principle this is not very surprising, as Aumann and Brandenburger (1995) provide sufficient, but not necessary conditions, for Nash equilibrium. However, it would be very interesting to study whether the fact that we obtain a Nash equilibrium in our example is a coincidence, or their result still holds after having relaxed their conditions.

Consider an asymmetric coordination game played by four players, Alice (a), Bob (b), Claire (c), and Donald (d), each of them choosing a pure strategies from $S_i = \{H, L\}$. If they all choose the same strategy they all receive a utility of 1, and otherwise they all receive 0, i.e., for each $i \in I$,

$$U_i(s_a, s_b, s_c, s_d) = \begin{cases} 2 & \text{if } s_i = H \text{ for all } i \in \{a, b, c, d\} \\ 1 & \text{if } s_i = L \text{ for all } i \in \{a, b, c, d\} \\ 0 & \text{otherwise} \end{cases}$$

Now, consider an epistemic model, $\mathcal{A}^{\Gamma}$, such that $\Omega = \{\omega_1, \omega_2, \omega_3\}$ is endowed with a uniform

common prior. The players' information partitions over the state space are

$$\begin{aligned}
\mathcal{P}_a &= \Big\{ \{\omega_1\}_L \ ; \ \{\omega_2, \omega_3\}_L \Big\} \\
\mathcal{P}_b &= \Big\{ \{\omega_1\}_L \ ; \ \{\omega_2\}_L \ ; \ \{\omega_3\}_H \Big\} \\
\mathcal{P}_c &= \Big\{ \{\omega_1, \omega_2\}_L \ ; \ \{\omega_3\}_H \Big\} \\
\mathcal{P}_d &= \Big\{ \{\omega_1\}_L \ ; \ \{\omega_2\}_L \ ; \ \{\omega_3\}_H \Big\}
\end{aligned}$$

with the indexes denoting the players' choice at each information set.

We make the following observations:

- ⊠ The conjectures are circular pairwise common knowledge at ω_1. To see this, consider the connected sequence of players $\{a, b, c, d\}$ and observe that Alice's and Bob's conjectures are pairwise common knowledge between them, i.e., it is pairwise common knowledge between Alice and Bob that (i) Alice is certain that everybody else plays L, and (ii) Bob is certain that everybody else plays L. Likewise, Bob's and Claire's conjectures are pairwise common knowledge between them, Claire's and Donald's conjectures are pairwise common knowledge between them, and finally Donald's and Alice's conjectures are pairwise common knowledge between them.
  However, the conjectures are not commonly known. In fact, they are not even mutually known, as Claire does not know Alice's conjecture at ω_1. More specifically, given her information set $P_c(\omega_1) = \{\omega_1, \omega_2\}$, Claire attaches probability 1/2 to "Alice being certain that everybody else chooses L", and probability 1/2 to "Alice being uncertain about whether everybody else chooses L or H". The latter implies that this example violates the at least one condition of Aumann and Brandenburger (1995).
- ⊠ Rationality is connected pairwise mutual knowledge at ω_1, as everybody knows that their immediate neighbors in the sequence $\{a, b, c, d\}$ are rational. However, it is not mutually known at ω_1 that everybody is rational, e.g., Claire does not know that Alice is rational at ω_1, as L is not a best response for Alice at ω_2. This implies that a second condition of Aumann and Brandenburger (1995) is violated.
- ⊠ For every $j \in I$, the rest of the players share the same marginal conjecture about j's choice at ω_1, in that $\text{marg}_{S_j} \mu_i(\omega_1) = \sigma_j$ for all $i \in I \setminus \{j\}$, where σ_j attaches probability 1 to "j playing L". Moreover, $(\sigma_a, \sigma_b, \sigma_c, \sigma_d)$ is a Nash equilibrium of our coordination game. Hence, both conclusions of Aumann and Brandenburger (1995) hold even though their two main assumptions fail.

As we have already mentioned at the beginning of this section, the preceding example raises the question whether there is a general connection between pairwise common knowledge of conjectures

and rationality on the one hand, and Nash equilibrium on the other. We address this point in the next section.

4. WEAK EPISTEMIC CONDITIONS NASH EQUILIBRIUM

The following result is a generalization of Theorem 1, in that it provides weaker sufficient conditions for Nash equilibrium. That is, we show that in fact common knowledge of the conjectures can be substituted by the weaker condition of circular pairwise common knowledge of the conjectures, and mutual knowledge of rationality can be substituted by connected pairwise mutual knowledge of rationality. Obviously, these conditions are technically — and in our view, also conceptually — much less demanding.

THEOREM 2 *Let Γ be a game and $\mathcal{A}^\Gamma$ be an epistemic model of it. Suppose that at some state $\omega \in \Omega$ the conjectures are circular pairwise common knowledge and rationality is connected pairwise mutual knowledge, i.e., for some strongly connected sequence of players $\{i_k\}_{k=1}^{m+1}$, there is some $(\mu_1, \ldots, \mu_N)$ such that*

$$\omega \in \Big(\bigcap_{k=1}^{m-1} K_{i_k,i_{k+1}}\big(R_{i_k} \cap R_{i_{k+1}}\big)\Big) \cap \Big(\bigcap_{k=1}^{m} CK_{i_k,i_{k+1}}\big([\mu_{i_k}] \cap [\mu_{i_{k+1}}]\big)\Big).$$

Then,

(i) $\operatorname{marg}_{S_j} \mu_i = \sigma_j$ *for all* $i \in I \setminus \{j\}$, *and*

(ii) $(\sigma_1, \ldots, \sigma_N)$ *is a Nash equilibrium of* Γ.

PROOF: Suppose for simplicity and without loss of generality that the connected sequence is $\{1, \ldots, n\}$, i.e., $i_k = k$ for all $k = 1, \ldots, n$.

Proof of (i). This result is a direct consequence of repeatedly applying Aumann's agreement theorem. First observe that

$$\begin{aligned} [\mu_i] &= \{\omega \in \omega : \mu_i(\omega) = \mu_i\} \\ &\subseteq \{\omega \in \Omega : \operatorname{marg}_{S_j} \mu_i(\omega) = \operatorname{marg}_{S_j} \mu_i\} \\ &=: [\operatorname{marg}_{S_j} \mu_i], \end{aligned}$$

implying that for $i, k \in I \setminus \{j\}$, if $\omega \in CK_{i,k}\big([\mu_i] \cap [\mu_k]\big)$ then $\omega \in CK_{i,k}\big([\operatorname{marg}_{S_j} \mu_i] \cap [\operatorname{marg}_{S_j} \mu_k]\big)$. Since, $\operatorname{marg}_{S_j} \mu_i$ and $\operatorname{marg}_{S_j} \mu_k$ are posterior probability distributions, in that for each $s_j \in S_j$,

$$\operatorname{marg}_{S_j} \mu_i(s_j) = \pi\big([s_j] \mid P_i(\omega)\big) \text{ and } \operatorname{marg}_{S_j} \mu_k(s_j) = \pi\big([s_j] \mid P_k(\omega)\big),$$

it follows from the agreement theorem Aumann (1976) that $\omega \in CK_{i,k}\big([\operatorname{marg}_{S_j} \mu_i] \cap [\operatorname{marg}_{S_j} \mu_k]\big)$ implies

$$\operatorname{marg}_{S_j} \mu_i = \operatorname{marg}_{S_j} \mu_k. \tag{1}$$

Fix a $j \in I$, and consider two arbitrary $i, k \in I \setminus \{j\}$. For simplicity, let $i < k$. We consider the three exhaustive and mutually exclusive cases:

(a) $i < k < j$: It follows from the previous step that $\omega \in CK_{i,i+1}\big([\mu_i] \cap [\mu_{i+1}]\big)$ implies $\operatorname{marg}_{S_j} \mu_i = \operatorname{marg}_{S_j} \mu_{i+1}$. inductively, we obtain $\operatorname{marg}_{S_j} \mu_i = \operatorname{marg}_{S_j} \mu_k$.

(b) $j < i < k$: This step is identical to (a).

(c) $i < j < k$: Similarly to the previous steps, we show that $\operatorname{marg}_{S_j} \mu_i = \operatorname{marg}_{S_j} \mu_1$, and likewise $\operatorname{marg}_{S_j} \mu_k = \operatorname{marg}_{S_j} \mu_m$. Finally, it follows from $\omega \in CK_{m,1}\big([\mu_m] \cap [\mu_1]\big)$ that $\operatorname{marg}_{S_j} \mu_m = \operatorname{marg}_{S_j} \mu_1$, which completes the proof.

Henceforth, for all $i, j \in I$, let $\sigma_j := \operatorname{marg}_{S_j} \mu_i$.

Proof of (ii). First, we show that for all $E \subseteq \Omega$,

$$\pi\big(E \cap [s_i] \;\big|\; P_i(\omega)\big) = \pi\big(E \;\big|\; P_i(\omega)\big) \cdot \pi\big([s_i] \;\big|\; P_i(\omega)\big). \tag{2}$$

This follows directly from the fact that $[s_i]$ is $\mathcal{P}_i$-measurable: More specifically, if $c_i(\omega) = s_i$, then $P_i(\omega) \subseteq [s_i]$ in which case $\pi\big(E \cap [s_i] \;\big|\; P_i(\omega)\big) = \pi\big(E \;\big|\; P_i(\omega)\big) = \pi\big(E \;\big|\; P_i(\omega)\big) \cdot \pi\big([s_i] \;\big|\; P_i(\omega)\big)$. Alternatively, if $c_i(\omega) \neq s_i$, then $P_i(\omega) \cap [s_i] = \emptyset$. Then, $\pi\big(E \cap [s_i] \;\big|\; P_i(\omega)\big) \leq \pi\big([s_i] \;\big|\; P_i(\omega)\big) = 0$.

Now, we show that for all $i \in I$

$$\mu_i(\omega) = \operatorname{marg}_{S_1} \mu_i(\omega) \times \cdots \times \operatorname{marg}_{S_{i-1}} \mu_i(\omega) \times \operatorname{marg}_{S_{i+1}} \mu_i(\omega) \times \cdots \times \operatorname{marg}_{S_n} \mu_i(\omega). \tag{3}$$

Without loss of generality, we show it for player 1. Observe that for an arbitrary $(s_2, \ldots, s_n) \in S_{-1}$,

$$\begin{aligned}
\mu_1(\omega)(s_2, \ldots, s_n) &= \pi\big([(s_2, \ldots, s_n)] \;\big|\; P_1(\omega)\big) \\
&= \pi\big([s_2] \cap \cdots \cap [s_n] \;\big|\; P_1(\omega)\big) \\
&= \pi\big([s_2] \cap \cdots \cap [s_n] \;\big|\; M_{1,2}(\omega)\big) \\
&= \sum_{P_2 \subseteq M_{1,2}(\omega)} \pi\big([s_2] \cap \cdots \cap [s_n] \;\big|\; P_2\big) \cdot \pi\big(P_2 \;\big|\; M_{1,2}(\omega)\big).
\end{aligned}$$

Then, it follows from Equation (2) that

$$\begin{aligned}
\mu_1(\omega)(s_2,\ldots,s_n) &= \sum_{P_2\subseteq M_{1,2}(\omega)} \pi\big([s_3]\cap\cdots\cap[s_n] \;\big|\; P_2\big)\cdot\pi\big([s_2] \;\big|\; P_2\big)\cdot\pi\big(P_2 \;\big|\; M_{1,2}(\omega)\big)\\
&= \sum_{P_2\subseteq M_{1,2}(\omega)} \pi\big([s_3]\cap\cdots\cap[s_n] \;\big|\; P_2(\omega)\big)\cdot\pi\big([s_2] \;\big|\; P_2\big)\cdot\pi\big(P_2 \;\big|\; M_{1,2}(\omega)\big)\\
&= \pi\big([s_3]\cap\cdots\cap[s_n] \;\big|\; P_2(\omega)\big)\sum_{P_2\subseteq M_{1,2}(\omega)} \pi\big([s_2] \;\big|\; P_2\big)\cdot\pi\big(P_2 \;\big|\; M_{1,2}(\omega)\big)\\
&= \pi\big([s_3]\cap\cdots\cap[s_n] \;\big|\; P_2(\omega)\big)\cdot\pi\big([s_2] \;\big|\; M_{1,2}(\omega)\big)\\
&= \pi\big([s_3]\cap\cdots\cap[s_n] \;\big|\; P_2(\omega)\big)\cdot\pi\big([s_2] \;\big|\; P_1(\omega)\big)\\
&= \mu_2(\omega)(s_3,\ldots,s_n)\cdot\mu_1(\omega)(s_2).
\end{aligned}$$

Repeat the previous step inductively to obtain

$$\mu_1(\omega)(s_2,\ldots,s_n) = \mu_1(\omega)(s_2)\cdots\mu_{n-1}(\omega)(s_n),$$

and recall that all players agree on the marginal conjectures — it follows from *(i)* — implying that

$$\mu_1(\omega)(s_2,\ldots,s_n) = \mu_1(\omega)(s_2)\cdots\mu_1(\omega)(s_n),$$

which proves Equation (3).

Since every player knows their own strategy, it follows that $\big(\mu_1(\omega),\ldots,\mu_n(\omega)\big) = (\mu_1,\ldots,\mu_n)$. Recall from *(i)* that

$$(\sigma_1,\ldots,\sigma_n) := \big(\mathrm{marg}_{S_1}\,\mu_2(\omega),\ldots,\mathrm{marg}_{S_n}\,\mu_1(\omega)\big).$$

For an arbitrary $i\in I$, we are going to show that $s_i\in BR_i(\sigma)$ for every $s_i\in \mathit{supp}(\sigma_i)$. For every $s_i\in\mathit{supp}(\sigma_i)$ observe that there is some $\omega'\in P_{i+1}(\omega)$ such that $c_i(\omega')=s_i$. Since $\omega\in K(R_i)$ it follows that $P_{i+1}(\omega)\subseteq R_i$, and therefore $c_i(\omega')\in BR_i\big(\mu_i(\omega')\big)$. Finally, it follows from $\omega\in CK_{i,i+1}\big([\mu_i]\big)$ that $\mu_i(\omega')=\mu_i(\omega)$, and therefore

$$\begin{aligned}
s_i &\in BR_i\big(\mu_i(\omega')\big)\\
&= BR_i\big(\mu_i(\omega)\big)\\
&= BR_i\big(\mathrm{marg}_{S_1}\,\mu_i(\omega)\times\cdots\times\mathrm{marg}_{S_{i-1}}\,\mu_i(\omega)\times\mathrm{marg}_{S_{i+1}}\,\mu_i(\omega)\times\cdots\times\mathrm{marg}_{S_n}\,\mu_i(\omega)\big)\\
&= BR_i(\sigma_1,\ldots,\sigma_n),
\end{aligned}$$

which completes the proof. *Q.E.D.*

The contribution of the previous result to the understanding of the epistemic foundations of Nash equilibrium is twofold.

- Firstly, we weaken the standard epistemic conditions of Aumann and Brandenburger (1995), in that we substitute the global conditions of common knowledge of conjectures and mutual knowledge of rationality with local ones expressed in terms of pairwise common knowledge and pairwise mutual knowledge respectively. This is particularly important especially when studying equilibria of large games, such for instance economies with many agents.
- Secondly, our result provides some further insight on the relationship between Nash equilibrium and common (mutual) knowledge of rationality, which has attracted the attention of many prominent game theorists over the past couple of decades. More specifically, for many years it was suspected that common knowledge of rationality was an essential element of Nash equilibrium. This view was challenged by Aumann and Brandenburger (1995) who required only mutual knowledge of rationality in order to obtain Nash equilibrium. However, Polak more recently observed that Aumann-Brandenburger's conditions actually imply common knowledge of rationality (Polak, 1999), thus restoring some of the initial confidence in the importance of common knowledge of rationality for Nash equilibrium.

  Recall that Polak's result relies on common knowledge of the conjectures and mutual knowledge of rationality, two assumptions that we do not impose. At this point, we naturally wonder whether our conditions also imply common knowledge of rationality. As it turns out, they do not. In fact, they do not even imply mutual knowledge of rationality. To see this, recall our example from Section 3, where Claire at ω_1 does not know that Alice is rational, even though all the conditions of Theorem 2 are satisfied, and therefore the marginal conjectures form a Nash equilibrium.

  We find the latter quite surprising, as, at least to our knowledge, this is the first result in the literature which provides sufficient epistemic conditions for a Nash equilibrium without mutual knowledge of rationality being necessarily present.

5. TIGHTNESS OF THE RESULT

In this section, we provide an example illustrating that we cannot dispense circular pairwise common knowledge of the conjectures, and substitute it with connected pairwise knowledge of the conjectures, and in this sense our condition is tight. In fact, we show that if the conjectures are connected pairwise common knowledge instead f circular pairwise common knowledge, two players may not even agree on the marginal conjectures they hold about a third player, implying that Theorem 2.i does not necessarily hold.

Consider a 3-players anti-coordination game, played by Alice (a), Bob (b) and Claire (c), each of

them choosing a pure strategy from $S_i = \{H, L\}$. If they all choose the same pure strategy, they all receive a utility of 0, and otherwise they all receive 1, i.e., for each $i \in I$,

$$U_i(s_a, s_b, s_c) = \begin{cases} 0 & \text{if } s_a = s_b = s_c \\ 1 & \text{otherwise} \end{cases}$$

Now, consider an epistemic model, $\mathcal{A}^\Gamma$, such that $\Omega = \{\omega_1, \omega_2\}$ is endowed with a uniform common prior. The players' information partitions over the state space are

$$\begin{aligned} \mathcal{P}_a &= \left\{ \{\omega_1\}_H \ ; \ \{\omega_2\}_H \right\} \\ \mathcal{P}_b &= \left\{ \{\omega_1\}_H \ ; \ \{\omega_2\}_L \right\} \\ \mathcal{P}_c &= \left\{ \{\omega_1, \omega_2\}_L \right\} \end{aligned}$$

with the indexes denoting the players' choice at each information set.

Now, consider the connected sequence $\{a, b, c\}$, and note that at every $\omega \in \Omega$ it is commonly known that everybody is rational. Moreover, observe that each player's conjecture is pairwise common knowledge between the player and his his neighbors, e.g., at ω_1, Alice conjectures that her opponents choose $(s_b, s_c) = (H, L)$ with probability 1 and this is pairwise common knowledge between her and Bob; Bob conjectures that her opponents choose $(s_a, s_c) = (H, L)$ and this is common knowledge; and finally Claire conjectures that her opponents choose $(s_a, s_b) = (H, H)$ with probability $1/2$ and $(s_a, s_b) = (H, L)$ with probability $1/2$, and this is also common knowledge. However, observe that the conjectures are not circular pairwise common knowledge. This follows from the fact that every Hamiltonian cycle would also contain an edge between Alice and Claire, who do not satisfy pairwise common knowledge of the conjectures.

Finally, observe that Theorem 2.*i* is violated, as Alice and Claire disagree on what the believe Bob plays, i.e., Alice is certain at ω_1 that Bob chooses H, whereas Claire attaches probability $1/2$ to the event that Bob plays H.

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